# Resolution of the directed Oberwolfach problem with cycles of equal length

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# Acknowledgements

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2. University of Ottawa;
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**The setting:** Consider a conference with 42 participants. To facilitate networking, the organizing committee decides to host 41 banquets. The banquet hall has 2 tables that each sits 21 people.

**The problem:** The organizing committee needs a set of 41 seating arrangements (one for each banquet) such that each participant is seated to the right of every other participants exactly once.

Is this possible?

# The complete symmetric digraph

## Definition

*The* **complete symmetric digraph**, *denoted* $K_n^*$, *is the digraph on* $n$ *vertices in which for every pair of distinct vertices* $x$ *and* $y$, *there are arcs* $(x, y)$ *and* $(y, x)$.

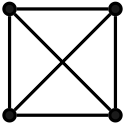

Figure: The complete graph $K_4$.

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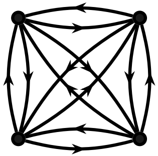

Figure: The complete symmetric digraph $K_4^*$.

# Cycle decomposition

## Definition

*A $\vec{C}_m$-**factor** of digraph $G$ is a spanning subdigraph of $G$ that is the disjoint union of directed m-cycles.*

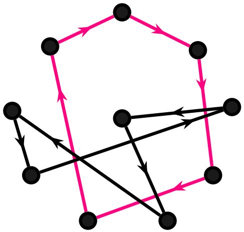

Figure: A $\vec{C}_5$-factor of $K^*_{10}$.

## Definition

*A $\vec{C}_m$-**factorization** (or **resolvable** $\vec{C}_m$-**decomposition**) of $G$, denoted R$\vec{C}_m$-D, is a decomposition of $G$ into $\vec{C}_m$-factors.*

**Question:** When does $K_n^*$ admit an R$\vec{C}_m$-D?

# The obvious necessary condition

**Necessary condition:** Let $\alpha$ be a positive integer. If the digraph $K_n^*$ admits an R$\vec{C}_m$-D, then $n = \alpha m$.

## Problem

*To find all integers $\alpha$ and $m$ for which $K_{\alpha m}^*$ admits an $R\vec{C}_m$-D.*

Observe that $\alpha =$number of cycles in a $\vec{C}_m$-factor.

In the language of our initial problem:

$m$ =number of participants seated at each table;
$\alpha$ =number of tables;
$\vec{C}_m$-factor = seating arrangement for 1 banquet.

A solution to our original problem is equivalent to a $\vec{C}_{21}$-factorization of $K^*_{2(21)}$.

## Small cases

The digraph $K^*_{\alpha m}$ admits an R$\vec{C}_m$-D when:

- $m = 3$ and $\alpha \neq 2$ (Bermond, Germa, and Sotteau, 1979)
- $m = 4$ and $\alpha \neq 1$ (Bennett and Zhan, 1990; Adams and Bryant, Unpublished)
- $m = 5$ and $\alpha \geqslant 103$ (Abel, Bennett, and Ge, 2002)

# Current results

**Theorem (Burgess and Šajna, 2014)**

*If $m$ is even or $\alpha$ is odd, then $K^*_{\alpha m}$ admits an $R\vec{C}_m$-D.*

We have a solution when tables are of even length or when we have an odd number of tables.

# Current results

What if we have an even number of tables of odd length?

**Theorem (Burgess and Šajna, 2014)**

*Suppose that $\alpha$ is an even integer and $m$ is an odd integer. If $K_{2m}^*$ admits an $R\vec{C}_m$-D, then $K_{\alpha m}^*$ also admits an $R\vec{C}_m$-D.*

It suffices to solve this problem when we have seating arrangements with two tables of odd length.

**Conjecture (Burgess and Šajna, 2014)**

*If $m$ is odd and $m \geqslant 5$, then $K_{2m}^*$ admits an $R\vec{C}_m$-D.*

# Current results

**Theorem (Burgess, Francetić, and Šajna, 2018)**

*If $m$ is odd and $5 \leqslant m \leqslant 49$, then $K^*_{2m}$ admits an $R\vec{C}_m$-D.*

**Theorem (Lacaze-Masmonteil and Šajna, unpublished)**

*If $m$ is divisible by $p \equiv 5 \pmod 6$, then $K^*_{2m}$ admits an $R\vec{C}_m$-D.*

# Result

**Theorem (Lacaze-Masmonteil, 2023+)**

*The digraph $K_{2m}^*$ admits an $R\vec{C}_m$-D for all odd $m \geqslant 11$.*

# Strategy

## Lemma (Burgess and Šajna, 2014)

*Let $\{G_1, G_2, \ldots, G_t\}$ be a decomposition of $K_n^*$ into spanning subdigraphs. If each $G_i$ admits a $R\vec{C}_m$-D, then $K_n^*$ admits an $R\vec{C}_m$-D.*

Step 1: Decompose $K_{2m}^*$ into $\frac{m-1}{2}$ spanning subdigraphs that fall into one of three isomorphisms classes: $G_1, G_2$ and $G_3$.

Step 2: Show that $G_1$, $G_2$, and $G_3$ admit an R$\vec{C}_m$-D .

# Decomposition of $K_{2m}^*$

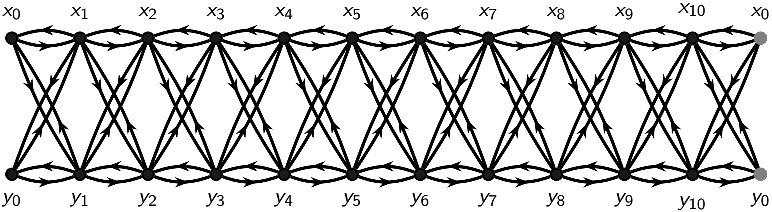


Figure: The digraph $G_1 = \vec{X}(\{\pm 1\}, 11) \wr \overline{K}_2$ that spans $K_{22}^*$.

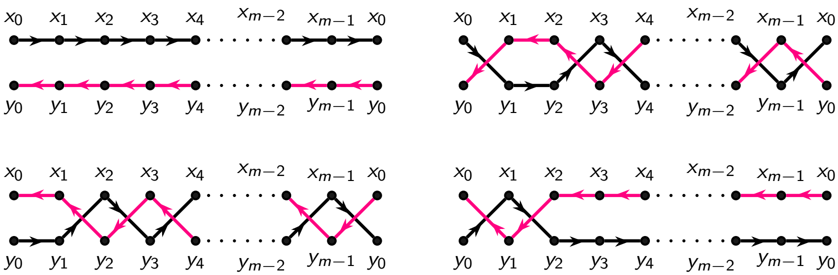


Figure: An R$\vec{C}_m$-D of $G_1$.

# Decomposition of $K_{2m}^*$

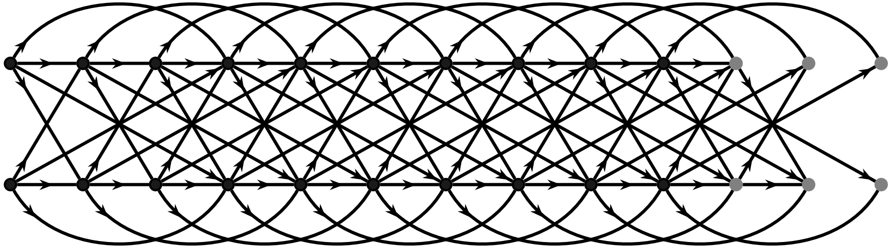

Figure: The spanning digraph $G_2 = \vec{X}(\{1,3\},11) \wr \overline{K}_2$ of $K_{2(11)}^*$.

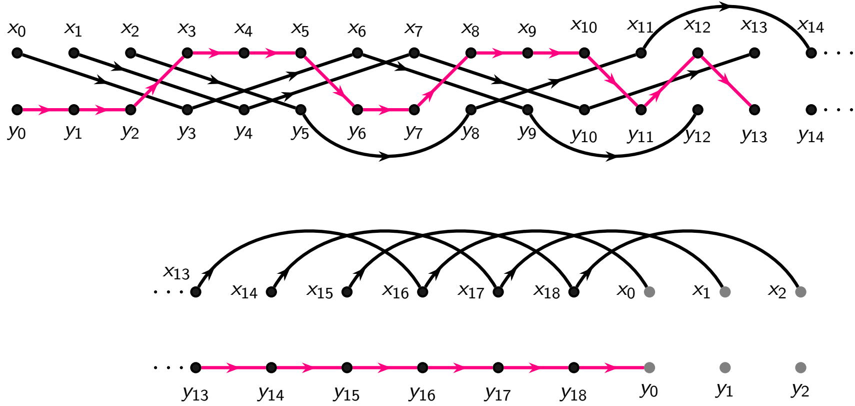


Figure: A $\vec{C}_{19}$ factor of $G_2$ when $m = 19$.

# Decomposition of $K_{2m}^*$

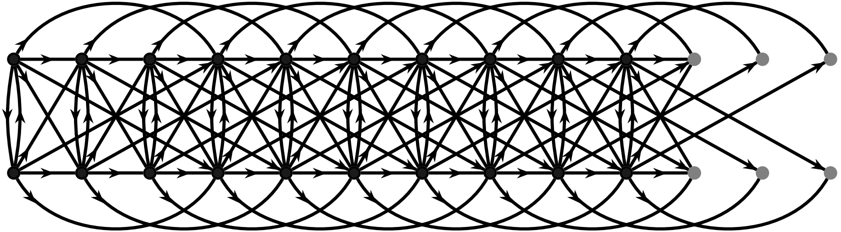

Figure: The spanning digraph $G_3 = \vec{X}(\{1,3\},11) \wr K_2^*$ of $K_{2(11)}^*$.

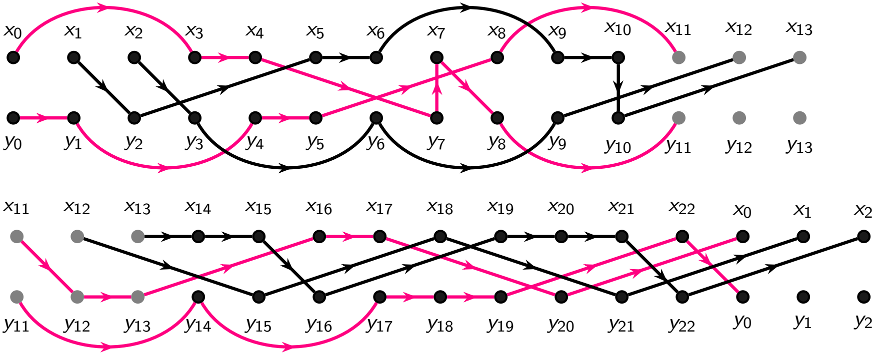


Figure: A $\vec{C}_{23}$-factor of $G_3$ for $m = 13$.

**Theorem**

*The digraph $K^*_{\alpha m}$ admits an $R\vec{C}_m$-D if and only if $(\alpha, m) \notin \{(1,6),(2,3),(1,4)\}$.*

The theorem above is a result of the work of: Bermond, Germa, and Sotteau (1979); Bennett and Zhan (1990); Adams and Bryant (Unpublished); Abel, Bennett, and Ge (2002); Burgess and Šajna (2014); Burgess, Francetić, and Šajna (2018); Lacaze-Masmonteil and Šajna (Unpublished); Lacaze-Masmonteil (2023+).